Certified Abstract Interpretation with Pretty-Big-Step Semantics

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Inria

12–14th of January

CPP'15

Previously at POPL

JSCert: A Trusted Mechanised JavaScript Specification

jscert.org

- An operational semantics for JavaScript;
- Trusted;
- *Huge* ($\sim$ 800 reduction rules).

How to derive an abstract interpreter from such a huge semantics?

... proven in Coq?

Let's make it correct *by construction*!

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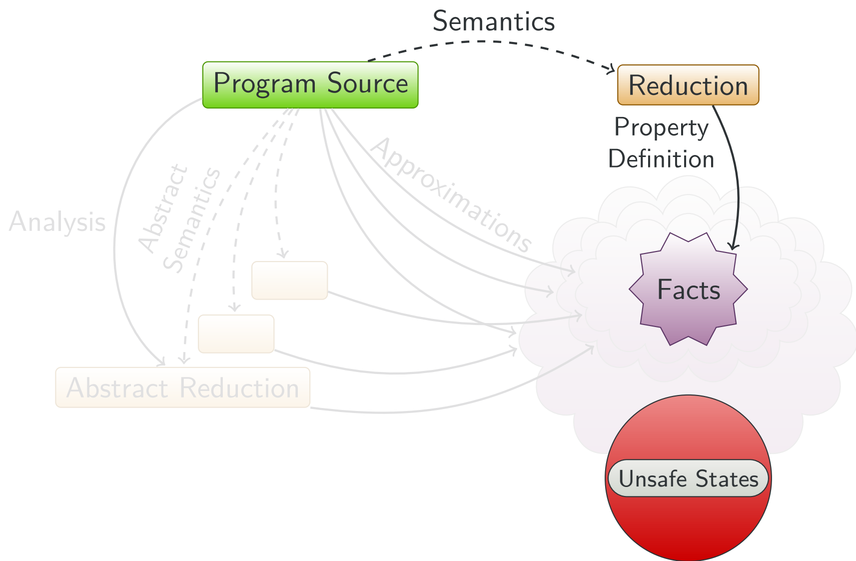
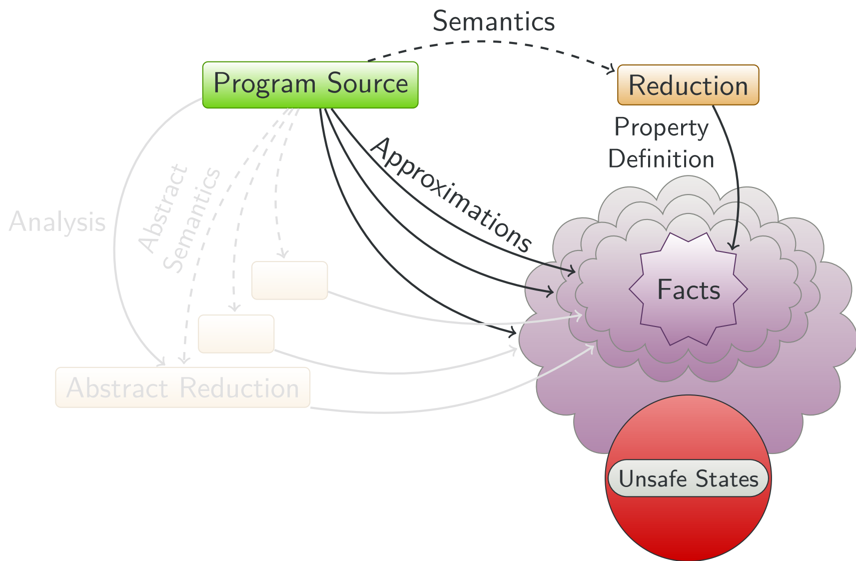
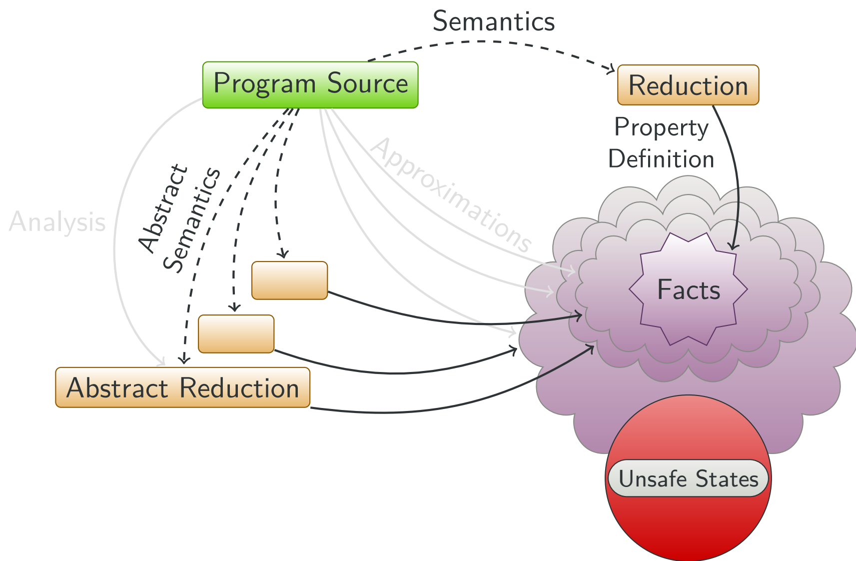
Abstract Interpretation




Semantics
Program Source
Reduction
Property
Definition
Approximations
Analysis
Abstract
Semantics
Facts
Abstract Reduction
Unsafe States

Abstract Interpretation



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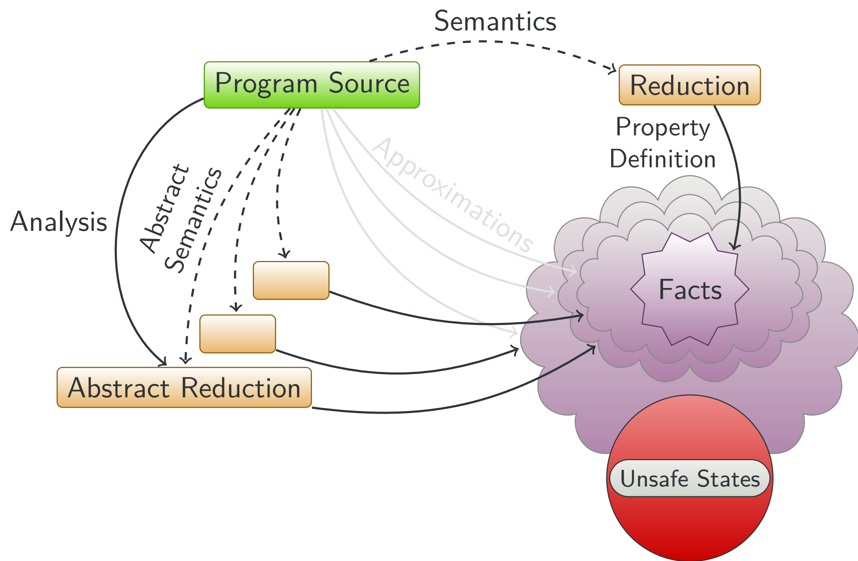

Concrete Domains

`int`, `bool`

Concrete Operations

`+`, `=`

Concrete Semantics

$t, \sigma \Downarrow r$

Abstract Domains

Sign

Abstract Operations

$+^\sharp$, $=^\sharp$

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$t, \sigma^\sharp \Downarrow^\sharp r^\sharp$

Abstract Interpreter

$f\left(t, \sigma^\sharp\right) = r^\sharp$

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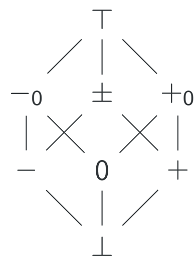
Defining Abstract Domains

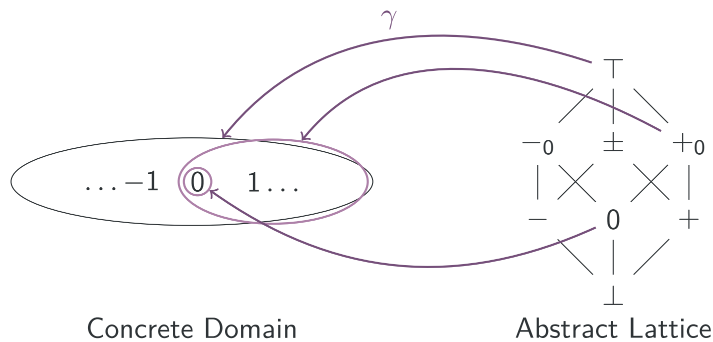
Concrete Domain



Abstract Lattice

The theory has already been formalized in Coq.

Defining Abstract Domains



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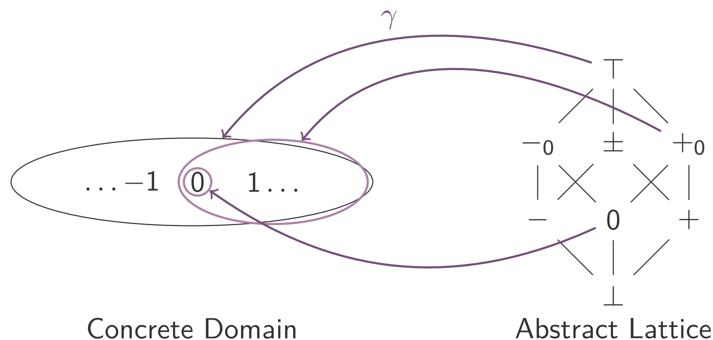

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Defining an Abstract Semantics, the Direct Approach

$$\textsc{IfTrue}\ \frac{e, E \Downarrow v \qquad s_1, E \Downarrow E'}{if\, e\, s_1\, s_2, E \Downarrow E'}\ v \neq 0 \qquad \textsc{IfFalse}\ \frac{e, E \Downarrow v \qquad s_2, E \Downarrow E'}{if\, e\, s_1\, s_2, E \Downarrow E'}\ v = 0$$

Let's just add $\sharp$ everywhere!

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Let's just add ♯ everywhere!

$$\textsc{IfAdHoc}\quad \frac{e, E^\sharp \Downarrow^\sharp v^\sharp \qquad s_1, E^\sharp \Downarrow^\sharp E_1^\sharp \qquad s_2, E^\sharp \Downarrow^\sharp E_2^\sharp}{if\, e\, s, E^\sharp \Downarrow^\sharp E_1^\sharp \sqcup E_2^\sharp} \quad v^\sharp = \top$$

1. Motivation
2. Pretty-Big-Step: a Generic Rule Format
3. Defining an Abstract Semantics Correct by Construction
4. Running Abstract Interpreters

Pretty-Big-Step

- Introduced by Chargéraud (ESOP 2013).
- Can be compiled from Small-Step (ESOP 2014).
- Similar to Big-Step semantics.
- But much more constrainted.

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$$\textsc{Rule1}\quad \frac{\mathfrak{u}_1, up(\sigma) \Downarrow r}{\mathfrak{l}, \sigma \Downarrow r}\ cond(\sigma)$$

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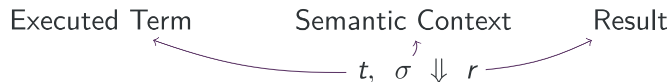

Each rule has

- A structural part: identifier, terms;
- A semantic part: side-conditions, transfer functions.

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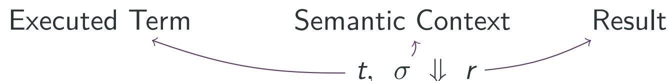

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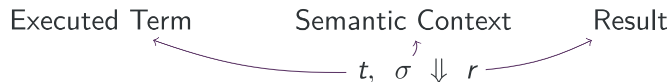

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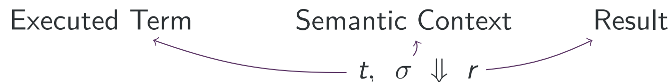

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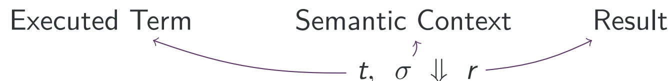

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Abstract Rules

Each rule has

- A structural part: identifier, terms;
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Abstract Rules

Shared between the concrete and abstract semantics

Each rule has

- A structural part: identifier, terms;
- A semantic part: side-conditions, transfer functions.

To be specified in the abstract semantics.
To be *locally* proved correct.

- The abstract semantics will follow the exact same structure as the concrete semantics.

Abstract Semantics

But we don't define $\Downarrow$ and $\Downarrow^\sharp$ the same way from the rules!

Concrete Semantics $\Downarrow$

$$\overline{}$$

At each step,
apply *one* rule that applies

Abstract Semantics $\Downarrow^\sharp$

$$\overline{}$$

At each step,
apply *all* the rules that apply

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Inductive interpretation of the rules $\Downarrow = \mathit{lfp}(\mathcal{F})$	Co-inductive interpretation of the rules $\Downarrow^\sharp = \mathit{gfp}\left(\mathcal{F}^\sharp\right)$

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	Allow approximations

Example of Concrete Rules

$$\textsc{While}(e,s)\quad \frac{\mathit{while}_1\ e\,s,\ \mathit{ret}\ E \Downarrow o}{\mathit{while}\ e\,s,\ E \Downarrow o}$$

$$\textsc{While1}(e,s)\quad \frac{e, E \Downarrow o \qquad \mathit{while}_2\ e\,s,\ (E,o) \Downarrow o'}{\mathit{while}_1\ e\,s,\ \mathit{ret}\ E \Downarrow o'}$$

$$\textsc{While2True}(e,s)\quad \frac{s, E \Downarrow o \qquad \mathit{while}_1\ e\,s,\ o \Downarrow o'}{\mathit{while}_2\ e\,s,\ (E, \mathit{val}\ v) \Downarrow o'}\quad v \neq 0$$

$$\textsc{While2False}(e,s)\quad \frac{}{\mathit{while}_2\ e\,s,\ (E, \mathit{val}\ v) \Downarrow \mathit{ret}\ E}\quad v = 0$$

Example of Abstract Rules

$$\textsc{While}(e, s)\quad \frac{\mathit{while}_1\ e\,s, E^\sharp \Downarrow^\sharp o^\sharp}{\mathit{while}\ e\,s, E^\sharp \Downarrow^\sharp o^\sharp}$$

$$\textsc{While1}(e, s)\quad \frac{e, E^\sharp \Downarrow^\sharp v^\sharp \qquad \mathit{while}_2\ e\,s, (E^\sharp, v^\sharp) \Downarrow^\sharp o^\sharp}{\mathit{while}_1\ e\,s, E^\sharp \Downarrow^\sharp o^\sharp}$$

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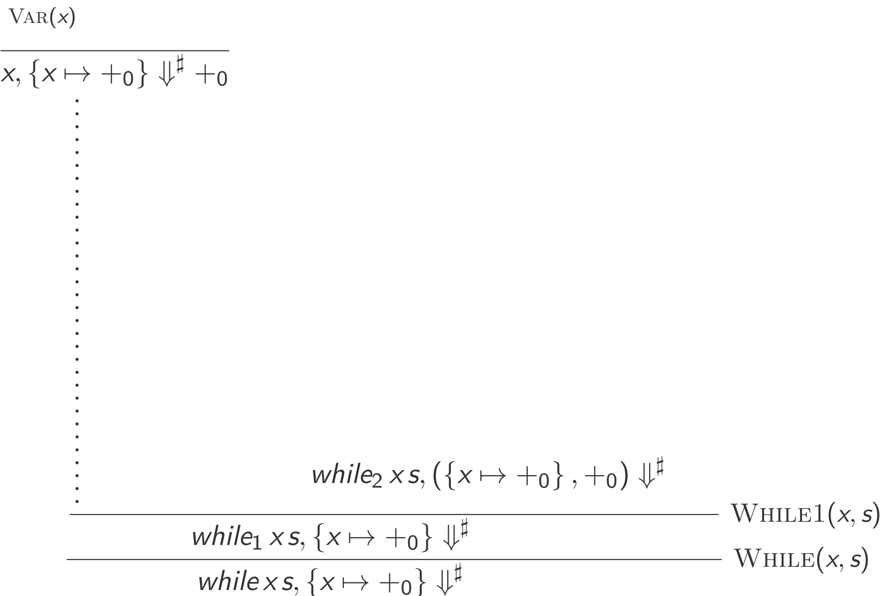
Example of Abstract Derivation Tree

$$\text{While1}(e, s)\ \frac{e, E \Downarrow o \qquad while_2\ e\, s, (E, o) \Downarrow o'}{while_1\ e\, s, ret\ E \Downarrow o'}$$

$$\dfrac{\dfrac{}{while_1\ x\, s, \{x \mapsto +_0\} \Downarrow^\sharp}\ \text{While1}(x, s)}{while\ x\, s, \{x \mapsto +_0\} \Downarrow^\sharp}\ \text{While}(x, s)$$

Example of Abstract Derivation Tree



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$$\dfrac{\textsc{Var}(x)}{x, \{x \mapsto +_0\} \Downarrow^\sharp +_0}$$

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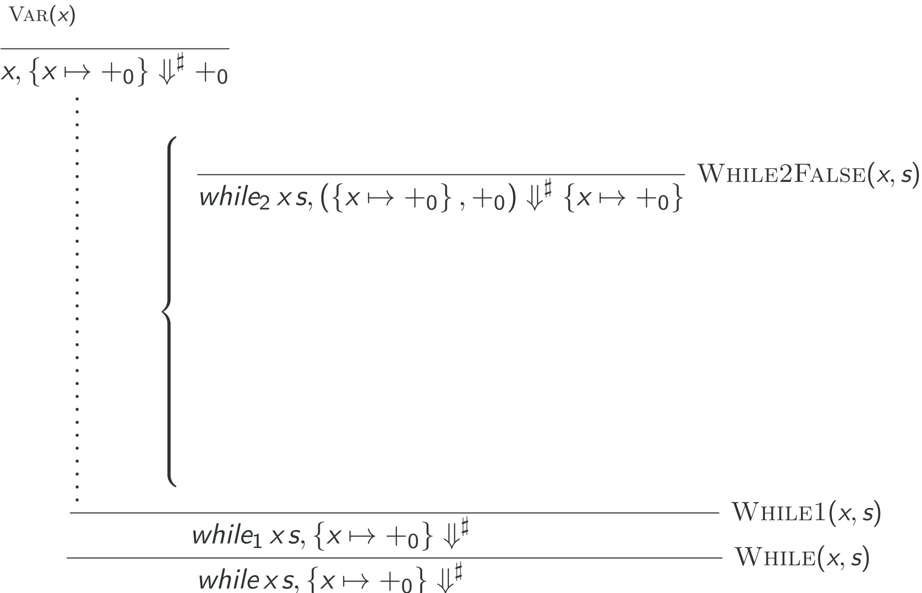

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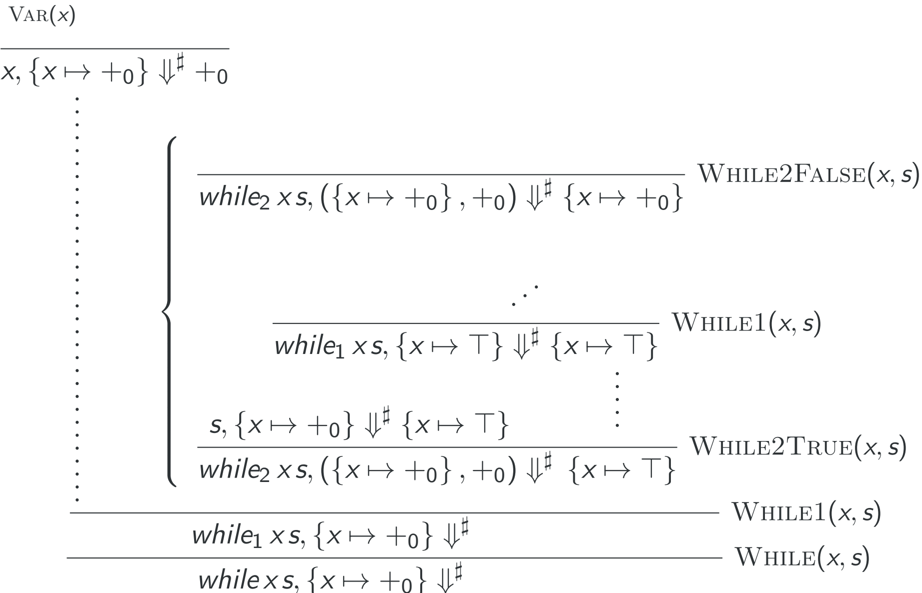

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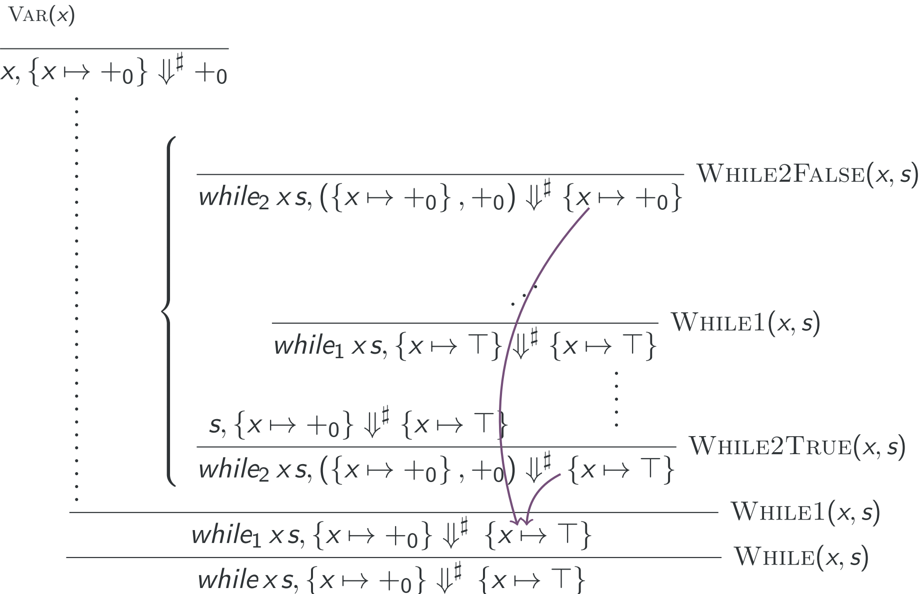

An Abstract Semantics Correct by Construction

Hypotheses:

- Correctness of the side-conditions,
- Correctness of the transfer functions.

Theorem (Correctness)

Let t a term, σ and $\sigma^\sharp$ a concrete and an abstract semantic contexts, and r and $r^\sharp$ a concrete and an abstract results.

$$\textit{If} \begin{cases} \sigma \in \gamma\left(\sigma^\sharp\right) \\ t, \sigma \Downarrow r \\ t, \sigma^\sharp \Downarrow^\sharp r^\sharp \end{cases} \quad \textit{then } r \in \gamma\left(r^\sharp\right).$$

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Proven independently of the rules!

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Defining Abstract Interpreters: a Verifier

- An abstract interpreter is a function building an abstract derivation.
- But this abstract semantic tree can be infinite!

A Verifier

- It takes an oracle, i.e., a set O of triples $t, \sigma^\sharp, r^\sharp$.

It tries to prove $O \subseteq {\mathcal{F}^\sharp}^+(O)$.
By PARK's principle, this implies $O \subseteq \Downarrow^\sharp$.

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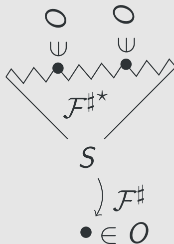

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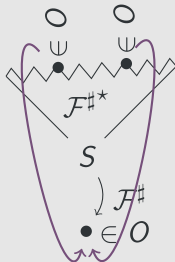

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Generic Abstract Interpreters

- We have built some *generic* abstract interpreters.
- We can extract them to OCaml and run them.

$$a := 6; b := 7; r := 0; n := a; \mathit{while}\, n\ (r := r + b; n := n - 1)$$

$$(\{r \mapsto +, b \mapsto +, a \mapsto +, n \mapsto \top\}, \bot)$$

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Conclusion and Future Works

We have investigated how to define, in Coq, certified abstract interpreters for pretty-big-step semantics.

Recipe

1. define the concrete semantics;
2. define the abstract domains and operations on the abstract domain,
   - this automatically defines an abstract semantics;
3. prove the abstract operations are correct,
   - this implies the abstract semantics is correct;
4. define an analysis.

Future Works

- Apply it to JSCert.
- Allow non-local reasonning.
- Taking into account non-terminating behaviours.

Thanks You for Listening!

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Bonus Slides

Concrete Semantics

$$
\begin{array}{l}
apply_i\,(\Downarrow_0) := \\
\quad match\ rule(i)\ \ with \\
\quad \mid\ \ Ax\,(ax) \qquad\quad \Rightarrow \{(\mathfrak{l}_i, \sigma, r) \mid ax(\sigma) = \texttt{Some}(r)\} \\
\quad \mid\ \ R_1\,(up) \qquad\quad \Rightarrow \left\{ (\mathfrak{l}_i, \sigma, r) \ \middle|\ \begin{array}{l} \quad up\,(\sigma) = \texttt{Some}(\sigma') \\ \wedge\, \mathfrak{u}_{1,i}, \sigma' \Downarrow_0 r \end{array} \right\} \\
\quad \mid\ \ R_2\,(up, next) \Rightarrow \left\{ (\mathfrak{l}_i, \sigma, r) \ \middle|\ \begin{array}{l} \quad up\,(\sigma) = \texttt{Some}(\sigma') \\ \wedge\, \mathfrak{u}_{2,i}, \sigma' \Downarrow_0 r_1 \\ \wedge\, next\,(\sigma, r_1) = \texttt{Some}(\sigma'') \\ \wedge\, \mathfrak{n}_{2,i}, \sigma'' \Downarrow_0 \texttt{Some}(r) \end{array} \right\}
\end{array}
$$

$\Downarrow = lfp\,(\mathcal{F})$

$\mathcal{F}\,(\Downarrow_0) = \{(t, \sigma, r) \mid \exists i, cond_i\,(\sigma) \wedge (t, \sigma, r) \in apply_i\,(\Downarrow_0)\}$

Abstract Semantics

$$apply_i^\sharp\left(\Downarrow_0^\sharp\right) = \left\{ (t, \sigma, r) \;\middle|\; \begin{array}{l} \exists \sigma_0, \exists r_0, \\ \quad \sigma \sqsubseteq^\sharp \sigma_0 \wedge r_0 \sqsubseteq^\sharp r \wedge \\ \quad (t, \sigma_0, r_0) \in apply_i\left(\Downarrow_0^\sharp\right) \end{array} \right\}$$

$\Downarrow^\sharp = gfp\left(\mathcal{F}^\sharp\right)$

$$\mathcal{F}^\sharp\left(\Downarrow_0^\sharp\right) = \left\{ (t, \sigma, r) \;\middle|\; \begin{array}{l} \forall i.\ t = \mathfrak{l}_i \Rightarrow cond_i(\sigma) \Rightarrow \\ \qquad (t, \sigma, r) \in apply_i^\sharp\left(\Downarrow_0^\sharp\right) \end{array} \right\}$$

Non Local Reasonning

$$if\,x\ (r := 0)\ (r := x)$$

Analysing in $\{x \mapsto +\}$

- Only the rule IfTrue applies.
- We get $r \mapsto 0$.

Analysing in $\{x \mapsto \top\}$

- Both rules IfTrue and IfFalse apply.
- We get $r \mapsto 0$ from IfTrue.
- We get $r \mapsto \top$ from IfFalse.
- We get $r \mapsto \top$ at the end.

```
CoInductive aeval : term -> ast -> ares -> Prop :=
  | aeval_cons : forall t sigma r,
      (forall n,
        t = left n ->
        acond n sigma ->
        aapply n sigma r) ->
      aeval t sigma r
with aapply : name -> ast -> ares -> Prop :=
  | aapply_cons : forall n sigma sigma' r r',
      sigma ⊑ sigma' ->
      r' ⊑ r ->
      aapply_step n sigma' r' ->
      aapply n sigma r
```

```
with aapply_step : name -> ast -> ares -> Prop :=
  | aapply_step_Ax : forall n ax sigma r,
      rule_struct n = Rule_struct_Ax _ ->
      arule n = Rule_Ax ax ->
      ax sigma = Some r ->
      aapply_step n sigma r
  | aapply_step_R1 : forall n t up sigma sigma' r,
      rule_struct n = Rule_struct_R1 t ->
      arule n = Rule_R1 _ up ->
      up sigma = Some sigma' ->
      aeval t sigma' r ->
      aapply_step n sigma r
  | aapply_step_R2 : forall n t1 t2 up next
                        sigma sigma1 sigma2 r r',
      rule_struct n = Rule_struct_R2 t1 t2 ->
      arule n = Rule_R2 up next ->
      up sigma = Some sigma1 ->
      aeval t1 sigma1 r ->
      next sigma r = Some sigma2 ->
      aeval t2 sigma2 r' ->
      aapply_step n sigma r'.
```

1. Motivation
2. Pretty-Big-Step: a Generic Rule Format
3. Defining an Abstract Semantics Correct by Construction
4. Running Abstract Interpreters